B. Sc. Part I (Hons.) 1st Paper

TRIGONOMETRY

GREGORY'S SERIES (contd.)

Gregory Series $\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - - - \text{to } \infty$,
θ lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$

or $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - - - \text{to } \infty$,
$|x| \leq 1$.

Sums

1. Prove that

$$\frac{\pi}{4} = \left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) - - - \text{to } \infty$$

Soln

$$RHS = \left(\frac{2}{3} - \frac{1}{3}\cdot\frac{2}{3^3} + \frac{1}{5}\cdot\frac{2}{3^5} - - - \text{to } \infty\right) + \left(\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{7^3} + \frac{1}{5}\cdot\frac{1}{7^5} - - \text{to } \infty\right)$$

$$\Rightarrow RHS = 2\left(\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{3}\right)^5 - - - \text{to } \infty\right) + \left(\frac{1}{7} - \frac{1}{3}\cdot\left(\frac{1}{7}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{7}\right)^5 - - - \text{to } \infty\right)$$

$$= 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

Use the formula $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$

$$\therefore RHS = \tan^{-1}\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2} + \tan^{-1}\frac{1}{7}$$

$\therefore RHS = \tan^{-1} \dfrac{\frac{2}{3}}{1-\frac{1}{9}} + \tan^{-1}\dfrac{1}{7}$

$= \tan^{-1} \dfrac{\cancel{2}}{\cancel{3}} \times \dfrac{\cancel{9}^{3}}{\cancel{8}_{4}} + \tan^{-1}\dfrac{1}{7}$

$= \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{1}{7}$

$\boxed{\text{Now we the formula } \tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}}$

$\therefore RHS = \tan^{-1}\dfrac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}$ $\cancel{= \tan^{-1}}$

$= \tan^{-1}\dfrac{\frac{21+4}{28}}{\frac{28-3}{28}} = \tan^{-1}\dfrac{\frac{25}{28}}{\frac{25}{28}}$

$= \tan^{-1} 1 = \tan^{-1}\tan\dfrac{\pi}{4}$

$= \dfrac{\pi}{4} = LHS$

Thus, LHS = RHS proved